LENTICULAR SPREADING OF UNDERGROUND WATERS

(RASTEKANIE LINZY GRUNTOVYKH VOD)

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We deal here with a transient problem which occurs in filtration theory, namely the spreading of a so-called lens in which there is fresh underground water, whilst the lens is located on the surface of underground soil waters of higher density (Fig. 1).

Solutions are derived for the plane and the axial-symmetrical problem; the following assumptions are introduced; the filtration coefficient k is the same for both regions; the density of the fresh water is ρ_0; the density of the salt water is ρ_1 and $\rho_0 < \rho_1$.

We are going to assume that in the plane parallel problem the filtration velocity in the lens in the direction of the x axis is considerably greater than that in the y axis direction, i.e. $v_x(0) \gg v_y(0)$; in the case of axial symmetry $v_r(0) \gg v_z(0)$.

Additionally the filtration velocities in the underlying strata occupied by the higher density subterranean water, on the surface of which the lens is situated, are considerably less than those within the lens. We therefore have:

For the plane problem

$$v_x{}^{(0)} \gg v_x{}^{(1)}, \qquad v_y{}^{(0)} \gg v_y{}^{(1)}$$

For the axial-symmetric problem

$$v_r{}^{(0)} \gg v_r{}^{(1)}, \qquad v_z{}^{(0)} \gg v_z{}^{(1)}$$

It can be concluded from the foregoing that the velocity potential within the lens, ϕ_0, will be a function of x (or r) only, whilst the velocity potential in the underlying region occupied by the subterranean water is zero.

In this case, with $\phi_1 = 0$ the velocities vanish, i.e.

$$v_x^{(1)} = \frac{\partial \varphi_1}{\partial x} = 0, \qquad v_y^{(1)} = \frac{\partial \varphi_1}{\partial y} = 0$$

With these assumptions the change in level of the underlying waters in the lower-layer due to the lens formation above, and its influence on the spreading is not taken into account. Besides it is assumed that underlying water of one density completely displaces underlying water of another density.

It follows from the above assumptions that in region *I*, at the outside boundary where it impinges on the outside medium (at point *C*)

$$p = 0, \qquad y = h(x) \tag{1}$$

Therefore, in view of the fact that

$$\varphi_0 = -k\left(\frac{p}{\rho_0 g} + y\right) \tag{2}$$

we have

$$\varphi_0 = -kh(x) \tag{3}$$

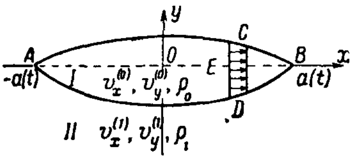


Fig. 1.

Let us determine the conditions at the boundary between regions *I* and *II*. In region *II* the potential $\phi = 0$; besides, we assume that the equation of the line which divides region *I* from *II* is of the form

$$y = -\alpha h(x) \tag{4}$$

Then the conditions on the boundary between regions *I* and *II* are

$$\varphi = -kh(x), \qquad y = -\alpha h(x) \tag{5}$$

The pressure at a point *D* on this boundary will be determined from the condition that $\phi_1 = 0$. From an expression similar to (2), assuming $\phi = 0$, we find

$$-k\left\{\frac{p}{\rho_1 g} + [-\alpha h(x)]\right\} = 0, \quad \text{or} \quad p = \rho_1 g \alpha h(x) \tag{6}$$

Because the pressure changes smoothly when going across the line *ADB*, in view of (2), we will have at point *D*

$$-kh(x) = -k\left[\frac{\rho_1 g \alpha h(x)}{\rho_0 g} - \alpha h(x)\right] \quad \text{or} \quad \left(\frac{\rho_1}{\rho_0} - 1\right)\alpha = 1 \tag{7}$$

Therefore the condition for the boundary between regions *I* and *II* will be fulfilled if

$$\alpha = \frac{\rho_0}{\rho_1 - \rho_0} \tag{8}$$

During the spreading therefore the ratio of the length *ED* to *EC* remains constant.

Thus the liquid which was above the x axis at the start always moves above this axis. The assumption that $\phi = -k(h)x$, leads us to the fact that $v_x = \partial\phi/\partial x = -kh'(x)$. In this case velocity v_x is independent of y, i.e. it is constant along a vertical segment.

Suppose, at the initial instant the lens has the shape shown on Fig. 1. The equations of lines *ACB* and *ADB* will be respectively

$$y = h(x), \qquad y = -\alpha h(x)$$

The total quantity of water enclosed by the lens will be (m is the porosity of the medium):

$$Q = m\left[\int_{-a}^{+a} h(x)\,dx + \alpha\int_{-a}^{+a} h(x)\,dx\right] = \frac{m\rho_1}{\rho_1 - \rho_0}\int_{-a}^{+a} h(x)\,dx \tag{9}$$

To study the motion of the lens it is sufficient to observe spreading of a given region occupied by underlying waters over an impermeable bottom surface.

This region (Fig. 2) will correspond to the portion of the underlying waters which is above the abscissa. We will assume that when

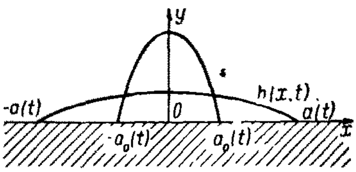


Fig. 2.

$$t = 0 \qquad h(x) = h_0(x) \qquad (-a_0 < x < a_0)$$

It is known that function $h(x, t)$ satisfies the Boussinesq equation [1]. For the one dimensional and axial-symmetric cases we have, respectively

$$\frac{\partial h}{\partial t} = \frac{k}{2m}\frac{\partial^2 h^2}{\partial x^2}, \qquad \frac{\partial h}{\partial t} = \frac{k}{2m}\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right)h^2 \tag{10}$$

We will give the solution to the problem of spreading of underground waters for the case where the quantity of fluid remains constant. Let us consider the one dimensional case. The first of the equations (10) in non-dimensional variables is as follows:

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u^2}{\partial \xi^2} \qquad \left(u = \frac{h}{h_0},\ \xi = \frac{x}{a_0}\right), \qquad \tau = \frac{kh_0}{2ma_0^2}\,t \tag{11}$$

$$u(\xi, \tau) = u_0(\xi) \qquad \text{for } \tau = 0$$

Here function $u_0(\xi)$ is given over intersect $-1 < \xi < 1$; outside this intersect it is zero. One can form the conclusion, evidently, that the required solution will tend to the solution which will prevail for impulsive starting conditions. This problem was dealt with by Barenblatt [2]. In this article these results are obtained by a different method in which several inaccuracies appearing in [2] are corrected.

We will look for a function $u(\xi)$ for which we have initial values

$$u_0(\xi) = 1 - \xi^2 \qquad \text{for } \tau = 0 \qquad (-1 < \xi < 1) \tag{12}$$

During motion we have

$$u = \beta(\tau)\,[\alpha^2(\tau) - \xi^2] \tag{13}$$

As the quantity of liquid within the lens remains constant, from the above we have

$$2\int_0^1 (1 - \xi^2)\,d\xi = 2\int_0^{\alpha(\tau)} \beta(\tau)\,[\alpha^2(\tau) - \xi^2]\,d\xi = \frac{4}{3}\beta(\tau)\,\alpha^3(\tau)$$

Because the integral in question must remain constant we have

$$\beta(\tau)\,\alpha^3(\tau) = 1, \quad \text{or} \quad \beta(\tau) = \frac{1}{\alpha^3(\tau)} \tag{14}$$

In accordance with (13) we have

$$u(\xi, \tau) = \frac{1}{\alpha^3(\tau)}\,[\alpha^2(\tau) - \xi^2] = \frac{1}{\alpha(\tau)} - \frac{\xi^2}{\alpha^3(\tau)} \tag{15}$$

Inserting the expression obtained into the initial equation (11) we find

$$\frac{\partial u}{\partial \tau} - \frac{\partial^2 u^2}{\partial \xi^2} = [4 - \alpha^2(\tau)\,\alpha'(\tau)]\left[\frac{1}{\alpha^4(\tau)} - \frac{3}{\alpha^6(\tau)}\xi^2\right] = 0$$

From this we have

$$4 - \alpha^2(\tau)\,\alpha'(\tau) = 0, \quad \text{or} \quad \alpha(\tau) = (C + 12\tau)^{\frac{1}{3}}$$

Because $\alpha(\tau) = 1$ when $\tau = 0$ we have

$$\alpha(\tau) = (1 + 12\tau)^{\frac{1}{3}}$$

(From (15) we have

$$u(\xi, \tau) = \frac{1}{1 + 12\tau}\left[(1 + 12\tau)^{\frac{2}{3}} - \xi^2\right] \tag{16}$$

On introducing the foregoing variables for the curve which bounds the upper part of the lens

$$h(x, t) = \frac{h_0}{1 + (6kh_0 / ma_0^2)\,t}\left[\left(1 + \frac{6kh_0}{ma_0^2}t\right)^{\frac{2}{3}} - \left(\frac{x}{a_0}\right)^2\right] \tag{17}$$

The equation of the curve which bounds the lower part of the lens, because of (4), is

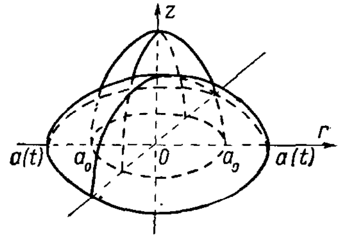
$$h_1(x, t) = -\frac{\rho_0}{\rho_1 - \rho_0}\,h(x, t) \tag{18}$$

The coordinate of the extreme point of the lens, located on the abscissa is

$$a(t) = a_0 \left(1 + \frac{6kh_0}{ma_0^2} t\right)^{\frac{1}{3}} \qquad (19)$$

Now let us consider the axially symmetrical case (Fig. 3). The fundamental differential equation in polar coordinates will be (10)

$$\frac{\partial u}{\partial \tau} = \left(\frac{\partial^2}{\partial \eta^2} + \frac{1}{\eta}\frac{\partial}{\partial \eta}\right) u^2, \quad \eta = \frac{r}{a_0} \qquad (20)$$



Фиг. 3

With $r = 0$ we have the initial condition $u_0 = 1 - \eta^2$ for the interval $-1 < \eta < +1$.

Let us assume that

$$u(\eta, \tau) = \beta(\tau)\,[\alpha^2(\tau) - \eta^2]$$

Maintenance of constant volume of liquid leads us to equations

$$\int_0^1 (1 - \eta^2)\, 2\pi\eta\, d\eta = \beta(\tau) \int_0^{\alpha(\tau)} [\alpha^2(\tau)\,\eta - \eta^3]\, 2\pi\eta\, d\eta, \qquad \beta(\tau)\,\alpha^4(\tau) = 1$$

from which $\beta(r) = a^{-4}(r)$; whence

$$u(\eta, \tau) = \frac{1}{\alpha^4(\tau)} - \frac{\eta^2}{\alpha^2(\tau)} \qquad (21)$$

Inserting this expression into Equation (20) we obtain

$$\frac{\partial u}{\partial \tau} - \left(\frac{\partial^2}{\partial \eta^2} + \frac{1}{\eta}\frac{\partial}{\partial \eta}\right) u^2 = [-4\alpha^3(\tau) + 16] \left[\frac{1}{2\alpha^6(\tau)} - \frac{\eta^2}{\alpha^8(\tau)}\right] = 0$$

The equation will be satisfied if

$$-\alpha^3(\tau)\,\alpha'(\tau) + 14 = 0, \qquad \alpha^4(\tau) = 16\tau + C \qquad (22)$$

Bearing in mind the initial condition $a(r) = 1$, for $r = 0$ we get

$$\alpha(\tau) = (1 + 16\tau)^{\frac{1}{4}}$$

Therefore

$$u(\tau, \eta) = \frac{1}{1 + 16\tau}\left[(1 + 16\tau)^{\frac{1}{2}} - \eta^2\right]$$

We now transform back to the previous variables and find an equation for the surface which bounds the upper part of the lens

$$h(t, r) = \frac{h_0}{1 + (8kh_0 / ma_0^2)\, t}\left[\left(1 + \frac{8kh_0}{ma_0^2} t\right)^{\frac{1}{2}} - \left(\frac{r}{a_0}\right)^2\right] \qquad (23)$$

The equation of the surface which bounds the lower part of the lens is of the form

$$h_1(t, r) = -\frac{\rho_0}{\rho_1 - \rho_0} h(t, r) \qquad (24)$$

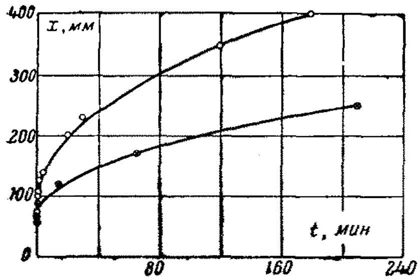


Fig. 4.

The coordinate of an extreme point of the lens will be

$$a(t) = a_0\left(1 + \frac{8kh_0}{ma_0^2} t\right)^{\frac{1}{4}} \qquad (25)$$

To confirm these results experiments were done with a slotted vessel (l = 1230 mm, b = 1.5 mm, a = 100 mm).

The denser liquid (ρ_1 = 1.225) was simulated in the first experiment by a mixture of glycerine and brine, 75 ml glycerine to 25 ml aqueous salt solution, the latter obtained by dissolving 30 gm cooking salt in 100 m^3 of fresh water.

In the second experiment the denser liquid (ρ_1 = 1.27) was simulated by salted glycerine, 100 ml glycerine to **10g cooking salt.**

The less dense liquid in the first experiment (ρ_0 = 1.18) was simulated by a tinted aqueous solution of glycerine, 25 ml fresh water to 75 m^3 glycerine. In the second experiment the less dense liquid (ρ_0 = 1.23) was simulated by tinted glycerine.

About 100 ml of the denser liquid was poured into the "slotted vessel". After its surface had levelled out, between 2 and 6 ml of the less dense liquid was poured on.

Results were worked out for two experiments and the law of displacement of the edge of the lens was established.

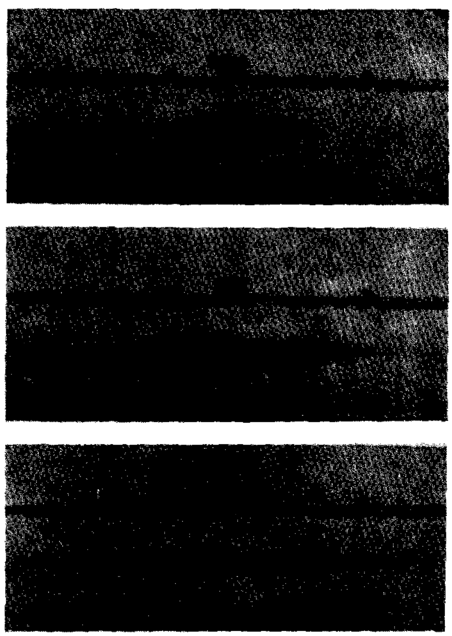


Fig. 5.

Figure 4 shows results worked out from formula (19); the upper curve corresponds to values ρ_0/ρ_1 = 0.964, the lower one ρ_0/ρ_1 = 0.969; experimental points are indicated. The agreement in both cases allows us to conclude that the initial assumptions, which form the basis of the solution, are confirmed.

Figure 5 shows photographs of three successive lens shapes which confirm the results of calculation qualitatively.

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